DISTRIBUTIONS OF CORANK 1
AND THEIR CHARACTERISTIC VECTOR FIELDS

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ABSTRACT. We prove that any 1-parameter family of corank 1 distributions (or Pfaff equations) on a compact manifold M^n is trivializable, i.e., transformable to a constant family by a family of diffeomorphisms, if all distributions of the family have the same characteristic line field. The characteristic line field is a field of tangent lines which is invariantly assigned to a corank one distribution. It is defined on M^n, if $n = 2k$, or on a subset of M^n called the Martinet hypersurface, if $n = 2k + 1$. Our second main result states that if two corank one distributions have the same characteristic line field and are close to each other, then they are equivalent via a diffeomorphism. This holds under a weak assumption on the singularities of the distributions. The second result implies that the abnormal curves of a distribution determine the equivalence class of the distribution, among distributions close to a given one.

0. INTRODUCTION

The well-known Gray theorem [G] states that any 1-parameter family of contact structures on a compact manifold M^{2k+1} is trivializable, i.e., transformable to a constant family by a family of diffeomorphisms. Our first main result generalizes this theorem to the case of "singular contact structures", for which the contact condition is satisfied on a dense subset of M, and to corank one distributions on manifolds of even dimension. In these cases the family of distributions has to preserve, when the parameter changes, a characteristic line field. The *characteristic line field* is a field of tangent lines which is invariantly assigned to a corank one distribution (it is defined on M^n if $n = 2k$, or on a hypersurface of M^n if $n = 2k+1$).

Our second main result states that if two corank one distributions have the same characteristic line field and are close to each other, then they are equivalent via a diffeomorphism. It means, in particular, that the characteristic line field contains complete information about the geometry of singularities of the distribution. Our results hold under a weak assumption on the distributions, called condition (A), saying that the depth of a characteristic ideal of the distribution is nondegenerate at singular points of the characteristic line field.

Let us state our second result in the simple case where the characteristic line field does not have singularities. Assume $n = 2k \geq 4$ and consider a smooth differential 1-form ω on M^n. Let $(\omega \wedge (d\omega)^{k-1})(p) \neq 0$ for all $p \in M^n$. Such a form ω defines the distribution $\Delta = \ker \omega$, called the *quasi-contact structure* defined by ω. Then $d\omega(p)$, restricted to the distribution $\Delta(p) = \ker \omega(p)$, is of maximal possible rank $2k - 2$ and has 1-dimensional kernel. We define the *characteristic line* at p by

$$L_p = \ker d\omega(p)|_{\Delta(p)}.$$

The *characteristic line field* $L = L(\omega)$ is the field of characteristic lines $p \to L_p$ on M. The following fact is a special case of our Theorem 1.2, the case where singularities are absent. (It is also a special case of a theorem in [MZh], Appendix A, concerning corank 1 distributions of constant class.)

Theorem 0.1. *Let Δ be the quasi-contact structure on a compact orientable manifold M^{2k} defined by a 1-form ω, and let $\tilde{\omega}$ be a 1-form such that $L(\tilde{\omega}) = L(\omega)$. If $\tilde{\omega}$ is sufficiently C^2-close to ω, then there exists a diffeomorphism of M^{2k} sending $\tilde{\Delta} = \ker \tilde{\omega}$ to Δ.*

If a compact orientable manifold M^{2k} admits a quasi-contact structure defined by a global 1-form ω, then its characteristic line field is generated by a nonvanishing global vector field and the Euler characteristic of M^{2k} is equal to zero. Thus a manifold M^{2k} with nonzero Euler characteristic admits only corank one distributions with singular characteristic line field. Even if M^{2k} admits a quasi-contact structure, singularities may appear naturally when restricting a corank one distribution to a submanifold of even dimension. Therefore, it is natural to ask whether Theorem 0.1 holds in the presence of singularities. Theorem 1.2 in Section 1 gives a positive answer under assumption (A), saying that the singularities of the characteristic vector field have a natural depth (and codimension, in the analytic case). This assumption excludes singularities of infinite codimension.

To state a similar result in the case $n = 2k + 1$, we introduce the set of points where ω does not satisfy the contact condition:

$$S = \{\, p \in M^{2k+1} \; : \; (\omega \wedge (d\omega)^k)(p) = 0\}.$$

This set is called the *Martinet hypersurface*. The Martinet hypersurface is the set of zeros of the function

$$H = \omega \wedge (d\omega)^k / \Omega,$$

where Ω is a volume form. If S is empty, i.e., ω is a contact 1-form on M^{2k+1}, then $\Delta = \ker \omega$ is globally equivalent to any distribution $\tilde{\Delta}$ sufficiently close to Δ. This follows from the theorem of Gray mentioned above. Assume now that S is nonempty. We shall call $\Delta = \ker \omega$ a *Martinet distribution* if it satisfies the following two conditions:

(a) $dH(p) \neq 0$ for all $p \in S$ (then S is smooth), and

(b) $\Delta_S = \ker \omega_S$ is a quasi-contact structure on S, where ω_S is the pullback of ω to S.

At each point of S we can define the *characteristic line*

$$L_p = \ker d\omega_S(p)|_{\ker \omega_S(p)}, \quad p \in S.$$

The *characteristic line field* $L = L(\omega)$ for a Martinet distribution Δ is the field of tangent lines $p \to L_p$ on S. It has no singularities. The following fact is a special case of our Theorem 1.4.

Theorem 0.2. *Let $\Delta = \ker \omega$ be a Martinet distribution on a compact orientable manifold M^{2k+1}. If the Martinet hypersurfaces and the characteristic line fields of Δ and $\tilde{\Delta} = \ker \tilde{\omega}$ are the same and $\tilde{\omega}$ is sufficiently C^3-close to ω, then there exists a diffeomorphism of the manifold sending $\tilde{\Delta}$ to Δ.*

Martinet distributions form a restrictive class of corank one distributions (L may have singularities). In particular, any Martinet distribution on M^3 has the Martinet hypersurface S, which consists of two-dimensional tori (each connected component of S has zero Euler characteristic, since S admits a 1-dimensional foliation defined by L). Our Theorem 1.4 will generalize Theorem 0.2 to the case of general corank one distributions and Pfaff equations on M^{2k+1}.

The appearance of the characteristic line field L as one of the main invariants of corank one distributions Δ has the following history. The first to study L was J. Martinet in [Mar] for the simplest occurring singularities of Δ on $\mathbb{R}^{2n+1}$ (when L has no singularities). J. Martinet also started to study typical singularities of L in the 3-dimensional case. These singularities were roughly classified in [JP], where the existence of a modulus in the classification of characteristic line fields was shown. It was proved in [Zh1] by obtaining a normal form for Δ that this modulus is the only invariant of Δ. This gave the complete local classification of germs of generic 2-distributions on 3-manifolds, with the characteristic line field as the complete invariant.

In the book [Zh2] the second author gave a classification of finitely determined singularities of corank one distributions and Pfaff equations on manifolds of any dimension. In this case again the characteristic line field L is a complete invariant. This justifies, to a large extent, making the following

Conjecture. *In the space of germs at $0 \in \mathbb{R}^n$ of corank one distributions on $\mathbb{R}^n$, there is an exceptional set of infinite codimension such that for any two distributions Δ_0 and Δ_1 away from this set, with the same characteristic line field L, there exists a local diffeomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ reducing Δ_1 to Δ_0.*

In a weaker form, with $n = 3$, this conjecture has already appeared in [JP] and in a letter from J. Martinet to the second author (in 1989). In this case the conjecture can be deduced from the results in [JZh2], [JZh1]. The exceptional set consists of germs that do not satisfy the assumption (A) or that do not have the property of zeros (see Section 1). In [JZh2] we proved that away from the exceptional set the restriction of the distribution to the Martinet hypersurface is a complete invariant for any $n = 2k + 1$. If $k = 1$, then the restriction can be identified with the characteristic line field.

The results of the present paper concern global corank one distributions close to a fixed distribution. They also deal with families of distributions. In this setting we eliminate some of the difficulties in the above conjecture which are due to non-close germs and the necessity of preserving a fixed point (the source of the germ).

Our transition to the global approach was inspired by the results in [Gol] and [MZh, Appendix A]. In [Gol] it is proved that a family of Engel structures E_t on a 4-manifold is trivializable provided that the characteristic line field of E_t does not depend on t. The result in Appendix A of [MZh] states that two close global corank one distributions of any constant class (in the Cartan-Frobenius sense) are diffeomorphic provided that they have diffeomorphic characteristic foliations. This is a generalization of the Gray theorem. All these results apply to objects without

singularities, whereas in the present paper we allow any singularities except certain ones of infinite codimension, excluded by the assumption (A). The presence of singularities leads to the main difficulties in our proofs. In Section 2 we explain that the assumption (A) is natural and give examples showing that it cannot be weakened.

In [JZh1] and [JZh2] we obtained local realization theorems (for germs at a fixed point), theorems characterizing the set of all possible characteristic line fields if $n = 3$. Combined with the reduction theorems, they lead to a number of applications including classification results. In this paper we leave aside the difficult task of obtaining global realization theorems. The absence of such theorems restricts, at present, the possibility of drawing immediate interesting conclusions concerning global classification of corank one distributions.

Note that the assumption of closeness which appears in Theorems 0.1 and 0.2 (and later in Theorems 1.2 and 1.4) is essential for our results. Already two contact structures that are far from each other are, in general, not equivalent. A classification of contact structures is known only on certain 3-dimensional manifolds, see e.g. [TEG]. We hope that our results can be used in global contact or quasi-contact geometry for studying singularities of 1-forms which appear when two global contact (or quasi-contact) structures are joined by a path.

There are natural consequences of our results concerning characteristic curves of a distribution, also called *singular curves* or *abnormal curves* in sub-Riemannian geometry, and the geometry of distributions (cf. [A], [BH], [LS], [Mon]). These curves coincide in our case with the integral curves of the characteristic line field.

Under the assumptions of Theorems 0.1 and 0.2, as well as those of Theorems 1.2 and 1.4 in Section 1, the singular curves of a corank 1 distribution determine the equivalence class of the distribution, among distributions close to a given one in the C^∞ topology.

1. Statement of results

We will deal with Pfaff equations, which are more general objects than (cooriented) corank one distributions. Let M^n denote a compact, orientable, Hausdorff manifold of dimension $n \geq 3$. By definition, a Pfaff equation is a set of differential 1-forms on M^n generated, as a module over the ring of functions, by a single 1-form ω. In other words, a Pfaff equation is a 1-form on M^n defined up to multiplication by a nonvanishing function. We denote the Pfaff equation by $P = (\omega)$. If ω vanishes at no points of M^n, then (ω) can be identified with the field of kernels of ω—a coorientable hyperplane field in TM^n. In general, a Pfaff equation is a more general object, since we do not exclude the possibility of ω vanishing at some points of the manifold.

All objects in this paper will belong to a fixed category which is either C^∞ or real analytic C^ω.

The case $n = 2k$. To any Pfaff equation P, and in particular to any cooriented corank one distribution, one can associate the *characteristic line field.*

Definition 1.1. If n is even, $n = 2k$, then any vector field X defined by the relation

$$X \rfloor \Omega = \omega \wedge (d\omega)^{k-1},$$

where Ω is any volume form, and ω any generator of P, is called a *characteristic vector field* of P. The line field on M^n generated by X, i.e., the mapping

$$p \to L_p = \{aX(p), \ a \in \mathbb{R}\} \subset T_p M^{2k},$$

is called the *characteristic line field*. The characteristic line field will be denoted by L or $L(\omega)$ or $L(P)$. The set $\{p \in M^{2k} : X(p) = 0\}$ is called the set of singular points of L and denoted by $Sing(L)$.

It is easy to check that any two characteristic vector fields differ by multiplication by a nonvanishing function, and consequently the characteristic line field is invariantly related to the Pfaff equation P, i.e., the choice of the generator ω of P and the volume form Ω is irrelevant. Note that if $(\omega \wedge (d\omega)^{k-1})(p) \neq 0$, then the definitions of L_p and L coincide with those given in the introduction.

In the presence of singularities we need an invariant that describes a "degree of degeneration" of singular points of a characteristic vector field X (for n odd it was introduced in [JZh2]). First we define an invariant that is slightly stronger than the set $Sing(L)$.

Let $p \in Sing(L)$. The *local characteristic ideal* I_p at a point $p \in M^{2k}$ of a Pfaff equation P is the ideal $I_p(X)$ in the ring of function germs at p, generated by the coefficients $a_1, \ldots, a_n$ of a characteristic vector field X of P, in some coordinate system near p. It is easy to see that the ideal I_p is invariantly related to the germ at p of $P = (\omega)$ (the choices of a characteristic vector field and of a local coordinate system are irrelevant). The germ at p of the set $Sing(L)$ is the zero set of I_p.

Definition 1.2. If $n = 2k$ and $p \in Sing(L)$, then we define $d_p(P) = d_p(X)$ as

$$d_p(P) = \text{depth} \, I_p(X).$$

Recall that the depth of a proper ideal $I \subset R$ of a ring R is the maximal length of a regular sequence of elements in I. A sequence $a_1, \ldots, a_r \in I$ is called regular if a_1 is not a zero divisor in R and, for any $i = 2, \ldots, r$, the element a_i is not a zero divisor in the quotient ring $R/(a_1, \ldots, a_{i-1})$, where $(a_1, \ldots, a_{i-1})$ denotes the ideal generated by $a_1, \ldots, a_{i-1}$. By definition, depth $R = \infty$.

Remark. In the analytic category, $d_p(P)$ is equal to the codimension in $\mathbb{C}^n$ of the germ at p of the set of complex zeros of the ideal I_p (i.e., the zero level set of the ideal generated by the complexification of the generators of I_p in some local coordinates). This follows from the fact that the complexification does not change the depth of an ideal of analytic function germs (cf. e.g. [E]) and from the equality of the depth(I) and the codimension of the analytic set of zeros of I for any ideal I of holomorphic function germs.

We introduce the following crucial condition:

(A) $\qquad\qquad\qquad\qquad d_p(P) \geq 3$ for any point $p \in Sing(L)$.

This condition is rather weak, in particular generic, as will be explained in Section 2.

The following theorems hold in the categories C^∞ and C^ω, with M a compact orientable manifold.

Theorem 1.1. *Let P_t, $t \in [0,1]$, be a family of Pfaff equations on M^{2k}, $k \geq 2$, that satisfies the following conditions.*
(a) All P_t define the same characteristic line field $L = L(P_t)$.

(b) All P_t satisfy condition (A).
Then there exists a family Φ_t of diffeomorphisms of M^{2k} sending P_t to P_0.

Theorem 1.2. *Let $P_0 = (\omega_0)$ and $P_1 = (\omega_1)$ be Pfaff equations on M^{2k}, $k \geq 2$, that have the same characteristic line field $L = L(P_0) = L(P_1)$. Assume that condition (A) holds for P_0. Then there exists a diffeomorphism Φ sending P_1 to P_0 provided that ω_1 is sufficiently close to ω_0 in the C^∞ topology.*

In the above theorems as well as in Theorems 1.3 and 1.4 below, all objects are in the same category C^∞ or C^ω, including regularity with respect to the parameter t. The diffeomorphism in Theorem 1.2 can be taken C^∞-close to the identity.

Remark (Closeness of ω_1 to ω_0)). In many cases one can present a number $r < \infty$ (depending on P_0) such that the closeness in the C^∞ topology in Theorem 1.2 can be replaced by closeness in the C^r topology. See Theorem 0.1 and Theorem B.2 in Appendix B.

The case $n = 2k + 1$**.** The most basic invariant of a Pfaff equation $P = (\omega)$ on M^{2k+1} is the set
$$S = \{p \in M^{2k+1} : \ (\omega \wedge (d\omega)^k)(p) = 0\},$$
called the *Martinet hypersurface*, which consists of points at which ω is not a contact form. This set, invariantly related to P, is the zero level of the function
$$H = \omega \wedge (d\omega)^k / \Omega,$$
where Ω is a volume form.

The ideal (H) of the ring of functions on M^{2k+1}, generated by H, is also invariantly related to P. It is called the *Martinet ideal*.

The characteristic line field of $P = (\omega)$ on M^{2k+1} is defined on the set S.

Definition 1.3. Any vector field X on M^{2k+1} satisfying the relation
$$X \rfloor \Omega = \omega \wedge (d\omega)^{k-1} \wedge dH \mod (H),$$
where ω is any generator of P, and H is any generator of the Martinet ideal, will be called a *characteristic vector field* of P. The line field on $S = \{H = 0\}$ defined by the relation
$$p \to L_p = \{aX(p), \ a \in \mathbb{R}\}, \qquad p \in S,$$
is called the *characteristic line field* of P. It will be denoted by L or $L(\omega)$ or $L(P)$. The set of singular points of L is defined as $Sing(L) = \{p \in M^{2k+1} : H(p) = 0, \ X(p) = 0\}$.

Above and in the rest of the paper the equality of two objects (vector fields, differential forms) mod (H) means that their difference is divisible over H in the space of objects of the same category. To check that the line field defined above is tangent to S, note that the definition of X implies that $X \rfloor dH = 0$ at any point of $S = \{H = 0\}$ and that X vanishes at any point of S at which the 1-form dH vanishes. Thus $X(p) \in T_pS$ at any point $p \in S$ at which S is a smooth hypersurface, and X vanishes at all other points. It is easy to check that the characteristic line field is invariantly related to P, i.e., the choices of the generator ω of P, the generator H of the Martinet ideal, and the volume form Ω are irrelevant.

Note that in the case of the Martinet singularity ($p \in S$ and $(\omega \wedge (d\omega)^{k-1} \wedge dH)(p) \neq 0$) the definitions of L_p and L coincide with those given in the introduction.

In order to deal with deeper singularities of P, namely those allowing singular points of L, we introduce our invariant $d_p(P)$ in the case of odd n as follows.

Let $p \in Sing(L)$. The *local characteristic ideal* I_p at a point $p \in M^{2k+1}$ of a Pfaff equation P is the ideal $I_p(H, X)$ in the ring of function germs at p generated by the germ H_p of a generator of the Martinet ideal and the coefficients $a_1, \ldots, a_n$ of a characteristic vector field X of P, in some coordinate system near p. The ideal I_p is invariantly related to the germ at p of $P = (\omega)$ (the choices of a characteristic vector field, a generator of the Martinet ideal, and of a local coordinate system are irrelevant). The germ at p of the set $Sing(L)$ is the zero set of I_p.

Definition 1.4. If $n = 2k + 1$ and $p \in Sing(L)$, then we define

$$d_p(P) = d_p(H, X)$$

as the maximal length of a regular sequence in the characteristic ideal $I_p(H, X)$, starting with the germ H_p as the first element.

Remarks. (a) In Noetherian rings all maximal regular sequences in I are of the same finite length. Moreover, any regular sequence can be completed to a maximal regular sequence. This implies that in the analytic category, independently of the parity of n, we have

$$d_p(P) = \mathrm{depth}(I_p).$$

(b) Similarly to the case $n = 2k$, in the analytic category, $d_p(P)$ is equal to the codimension in $\mathbb{C}^n$ of the germ at p of the set of complex zeros of the complexification of the ideal I_p.

To formulate our reduction theorem for the most general case, we need two properties of the Martinet ideal (H): the property of zeros and the extension property.

Definition 1.5. The Martinet ideal (H) has the *property of zeros* if for any $p \in S = \{H = 0\}$ the ideal in the ring of all function germs at p generated by the germ H_p of H at p coincides with the ideal in the same ring consisting of function germs vanishing on the germ at p of the set $S = \{H = 0\}$.

The property of zeros allows us to identify the Martinet hypersurface $S = \{H = 0\}$ with the Martinet ideal. In the case of germs this follows from the definition. Examples where the property of zeros is violated at a point p include:

(a) $H_p = H_1^2 H_2$, where H_1, H_2 are function germs and $H_1(p) = 0$;

(b) H_p is equivalent to $r^2 = x_1^2 + \cdots + x_n^2$;

(c) H_p is a flat germ (i.e., the Taylor series of H at p is zero);

(d) H_p is a zero divisor in the ring of all germs at p.

In case (c) the property of zeros is violated, since $\tilde{H}_p = r^{-2} H_p$ is smooth and has the same germ of zeros as H_p but $\tilde{H}_p \notin (H_p)$. Note that (d) is a particular case of (c).

The local version of the property of zeros (Definition 1.5) implies the global version: if a function f on M vanishes on the set $S = \{H = 0\}$, then f belongs to the ideal (H). This follows from the fact that division by H or by the germ H_p is unique (by (d) the germ H_p is not a zero divisor).[1]

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[1] In the C^∞-category the global and local versions of the property of zeros are equivalent (this follows from the partition of unity). In the real analytic category they are equivalent provided that the sheaf of functions vanishing on $S = \{H = 0\}$ is coherent. In the proof "global implies

We also need the *extension property* of the Martinet hypersurface $S = \{H = 0\}$. Denote by $C^\infty(M)$ the Fréchet space of smooth functions on M, equipped with the topology of convergence together with all derivatives. Let $C^\infty(M, S)$ denote its closed subspace of functions that vanish on S. We define the space of smooth functions on S as the quotient Fréchet space $C^\infty(S) = C^\infty(M)/C^\infty(M, S)$.

Definition 1.6. We say that S has the *extension property* if there exists a continuous linear operator $\lambda : C^\infty(S) \to C^\infty(M)$ such that $\lambda(f)|_S = f$ for all $f \in C^\infty(S)$.

The extension property automatically holds in the C^ω category, since it holds for any analytic subset S of M (see [BS] for a more general extension theorem). It also holds if we assume that (H) has, locally around any point $p \in S$, a generator that is analytic in some coordinate system.

The following theorems hold in the categories C^∞ and C^ω, with M compact and orientable.

Theorem 1.3. *Let P_t, $t \in [0, 1]$, be a family of Pfaff equations on M^{2k+1}, $k \geq 1$, that satisfies the following conditions.*

(a) All P_t have the same Martinet hypersurface S, which has the extension property, and their Martinet ideals have the property of zeros (and consequently are the same).

(b) All P_t define the same characteristic line field $L = L(P_t)$.

(c) All P_t satisfy condition (A).

Then there exists a family Φ_t of diffeomorphisms of M^{2k+1} sending P_t to P_0.

Remark. Recall that the extension property of S holds automatically in the C^ω category. We conjecture that in the C^∞ category the extension property in Theorem 1.3 also can be omitted. Our proofs show that this is so if the family P_t has a generator ω_t that is polynomial in t.

Theorem 1.4. *Let $P_0 = (\omega_0)$ and $P_1 = (\omega_1)$ be Pfaff equations on M^{2k+1}, $k \geq 1$, which have the same Martinet hypersurface $S = S(P_0) = S(P_1)$ and the same characteristic line field $L = L(P_0) = L(P_1)$. Assume that the Martinet ideal of P_0 has the property of zeros and P_0 satisfies condition (A). Then there exists a diffeomorphism Φ sending P_1 to P_0, provided that ω_1 is sufficiently close to ω_0 in the C^∞ topology.*

Remark (Closeness of ω_1 to ω_0). As in the even-dimensional case, often one can present a number $r < \infty$ (depending on P_0) such that the closeness in the C^∞ topology in Theorem 1.4 can be replaced by closeness in the C^r topology. See Theorem 0.2 and Theorem B.2 in Appendix B.

The contents of the further sections. In Section 2 we explain why the condition (A) is natural and give examples showing that it cannot be weakened. The consequences of condition (A) are explained in Section 3 and Appendix A: the condition (A) implies certain global division properties of a characteristic vector field. Section 4 contains auxiliary algebraic statements, which also will be used throughout the proofs. Using the division properties and these algebraic statements, we prove Theorems 1.1 and 1.3 in Sections 5 and 7, respectively. The proofs of these theorems are based on the homotopy method, according to which it suffices to prove

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local" one should use Cartan's Theorem A in [C], which says that any local section of a coherent analytic module belongs to the module generated by global sections.

the solvability of the equation

$$\text{(HE)} \qquad L_{Z_t}\omega_t + h_t\omega_t + \frac{d\omega_t}{dt} = 0$$

with respect to a family of vector fields Z_t and a family of functions h_t (here $L_Z\omega$ denotes the Lie derivative of ω along Z). Then the family Φ_t of diffeomorphisms obtained by integrating the family of vector fields Z_t,

$$\frac{d\Phi_t}{dt} = Z_t(\Phi_t), \qquad \Phi_0 = id,$$

transforms the Pfaff equations (ω_t) into (ω_0):

$$\Phi_t^*\omega_t = \psi_t\omega_0,$$

where $\psi_t = \exp(-\int_0^t \tilde{h}_s ds)$ and $\tilde{h}_t = h_t \circ \Phi_t$. In what follows the equation (HE) will be called the *homotopy equation* or *homological equation*.

Theorems 1.2 and 1.4 are proved in Sections 6 and 8 by reduction to Theorems 1.1 and 1.3. In these sections we show that if Pfaff equations P_0 and P_1 satisfy the assumptions of Theorem 1.2 or 1.4, then there exist generators ω_0 of P_0 and ω_1 of P_1 such that the path of Pfaff equations P_t generated by $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ satisfies the assumptions of Theorem 1.1 or Theorem 1.3.

In Appendix B we present certain topological properties of linear operators related to the Martinet ideal and the characteristic ideal. We also show a way of transition from the assumption of C^∞-closeness of ω_1 to ω_0 in Theorems 1.2 and 1.4 to the C^r-closeness with a certain $r < \infty$. In the simplest cases this way leads to Theorems 0.1 and 0.2 in the Introduction.

2. NECESSITY OF CONDITION (A)

In this section we explain why the condition

$$\text{(A)} \qquad d_p(P) \geq 3$$

is natural, and we give examples showing that this condition cannot be weakened: if depth $d_p(P) = 2$, then our theorems are not true anymore.

Fix a point $p \in M^n$ and denote by J_p^i the space of i-jets of 1-forms at p. The condition that p is a singular point of the characteristic line field L, i.e. $p \in Sing(L)$, is the condition

$$(\omega \wedge (d\omega)^{k-1})(p) = 0,$$

if $n = 2k$, and

$$(\omega \wedge (d\omega)^k)(p) = 0, \qquad (\omega \wedge (d\omega)^{k-1} \wedge dH)(p) = 0,$$

if $n = 2k + 1$. It involves the i-jet at p of a generator ω of P, where $i = 1$ if n is even and $i = 2$ if n is odd. This condition distinguishes a certain subset of J_p^i—the space of i-jets of ω at p. It is not difficult to see that for any parity of n this subset is a stratified submanifold of codimension 3 (see [Mar], [Zh2], or [JZh2] for more details). Consequently, for generic ω the set $Sing(L)$ is either empty or a submanifold of M^n of codimension 3. In the real analytic category (and conjecturally, in the smooth category too) the set of 1-forms ω violating (A) has infinite codimension in the space of all 1-forms on M^n; see [JZh2], Proposition 3.4 and Theorem A2 (Appendix 2).

The following examples show that the condition (A) cannot be replaced by the condition $d_p(P) \geq 2$. In these examples $\dim M = 4$ and $\dim M = 5$. They can be

extended to higher dimensions. We have not found an example in the 3-dimensional case, but we believe that such an example exists.

Example 1. Consider the family of Pfaff equations on the 4-torus T^4 generated by 1-forms

$$\omega_t = d\theta_1 + (\sin\theta_3 \sin\theta_4 + t)d\theta_2.$$

The characteristic vector field X_t is the same for all t:

$$X_t = X_0 = \cos\theta_4 \sin\theta_3 \frac{\partial}{\partial\theta_3} - \cos\theta_3 \sin\theta_4 \frac{\partial}{\partial\theta_4}.$$

The set S of singular points of X_0 is the union of 8 disjoint 2-dimensional tori (4 of them are described by the equations $\theta_3, \theta_4 \in \{\pi/2, -\pi/2\}$, and the other 4 by the equations $\theta_3, \theta_4 \in \{0, -\pi\}$). The codimension of S is 2; therefore depth $I_p = 2$. The restriction of (ω_t) to any of these 2-dimensional tori is a Pfaff equation generated by a 1-form $\alpha_t = d\theta_1 + (\delta + t)d\theta_2$, where $\delta \in \{0, \pm1\}$ depending on the torus. This Pfaff equation can be identified with the vector field $V_t : (\delta + t)\partial/\partial\theta_1 - \partial/\partial\theta_2$ defined up to multiplication by a nonvanishing function. It follows that the phase portrait of V_t on the torus is invariantly related to (ω_t). It is well known that the equivalence of the phase portraits of V_{t_1} and V_{t_2} with a fixed δ implies $t_1 = t_2$ provided that t_2 is close to t_1; see [ArIl] (the parameter t corresponds to the rotation number). Therefore the parameter t of the family P_t is a modulus (a parameter varying continuously and distinguishing nonequivalent Pfaff equations).

Example 2. Consider the family of Pfaff equations on the 5-torus

$$T^5(\theta_1, \theta_2, \phi_1, \phi_2, \phi_3)$$

generated by 1-forms

$$\omega_t = (A(\theta_1, \theta_2) + B_t(\theta_1, \theta_2, \phi_2))\, d\phi_1 + C(\theta_1, \theta_2)d\phi_2 + d\phi_3,$$

where

$$A(\theta_1, \theta_2) = 3(\sin\theta_1 + \sin\theta_2),$$
$$B_t(\theta_1, \theta_2) = t\sin\phi_2 \left(1 - \cos(\theta_1 - \theta_2)\right),$$
$$C(\theta_1, \theta_2) = \cos\theta_1 + \cos\theta_2.$$

A simple calculation gives that $\omega_t \wedge (d\omega_t)^2 = \sin(\theta_1 - \theta_2) \cdot Q_t \cdot \Omega$, where Ω is a volume form and Q_t is a family of nonvanishing functions on T^5, if $t \in [-1, 1]$. Therefore the Martinet ideal is the same for all t; it is generated by the function $\sin(\theta_1 - \theta_2)$. The Martinet hypersurface consists of two disjoint 4-tori:

$$S = T_1^4 \cup T_2^4, \quad T_1^4 = \{\theta_2 = \theta_1\}, \quad T_2^4 = \{\theta_2 = \theta_1 + \pi\}.$$

Since the function B_t vanishes on the torus T_1^4, the restriction of (ω_t) to T_1^4 does not depend on t. The restriction of (ω_t) to the torus $T_2^4(\phi_1, \phi_2, \phi_3, \theta_1)$ depends on t; it is the Pfaff equation (α_t), where $(\alpha_t) = 2t\sin\phi_2 d\phi_1 + d\phi_3$. The characteristic vector field of (ω_t) restricted to T_2^4 is the characteristic vector field of (α_t). It is $2t\cos\phi_2 \partial/\partial\theta_1$. Assume that $t \neq 0$. Then the characteristic line field does not depend on t. The set of its singular points is the union of two disjoint 3-tori $T_\pm^3$, given by the equations $\phi_2 = \pm\pi/2$. The restriction of (ω_t) to $T_\pm^3$ (or, the same, the restriction of (α_t) to $T_\pm^3$) is the Pfaff equation of the form (β_t), $\beta_t = 2t\delta d\phi_1 + d\phi_3$, $\delta \in \{-1, 1\}$. Consider the vector field $V_t : -\partial/\partial\phi_1 + 2t\delta\partial/\partial\phi_3$ on the 2-torus $T^2 = T^2(\phi_1, \phi_3)$. It is easy to see that the Pfaff equations (β_{t_1}) and (β_{t_2}) on the

3-torus are equivalent if and only if the phase portraits of V_{t_1} and V_{t_2} on the 2-torus are equivalent. As in the previous example, this is so if and only if $t_1 = t_2$ provided that t_2 is close to t_1. Therefore the parameter t of the family (ω_t) of Pfaff equations is a modulus, although these Pfaff equations have the same Martinet ideal (satisfying the property of zeros) and the same characteristic line field. The reason for that is the violation of the assumption (A)—the depth of the characteristic ideal is equal to 2 instead of 3.

3. Condition (A) and division properties

In this section we explain implications of condition (A) which will be essential in further proofs. The main implications are the following global division properties of a characteristic vector field X. As before, we work in the C^∞ and C^ω categories.

Proposition 3.1.a. *If a Pfaff equation $P = (\omega)$ satisfies condition (A), then any characteristic vector field X of P has the following division properties.*

(i) If n is even, then for any vector field Y and any r-form ν on M^n with $r = n - 1$ or $r = n - 2$ the equality

$$X \rfloor \nu = 0 \quad implies \quad \nu = X \rfloor \mu,$$

for an $(r + 1)$-form μ on M^n, and the equality

$$X \wedge Y = 0 \quad implies \quad Y = fX$$

for a function f on M^n.

(ii) If n is odd, then for any vector field Y on M^n and any $(n - 1)$-form ν on M^n the equality

$$X \rfloor \nu = 0 \mod (H) \quad implies \quad \nu = X \rfloor \mu \mod (H),$$

for an n-form μ on M^n, and the equality

$$X \wedge Y = 0 \mod (H) \quad implies \quad Y = fX \mod (H)$$

for a function f on M^n. Here (H) is the Martinet ideal of P, and we assume that (H) has the property of zeros.

This proposition is a corollary of a general theorem in [DJ] on division properties of the interior product with a section X of a vector bundle (see Appendix A for the proof).

We also need a division property with parameters. In the next and all further statements in this section a 1-parameter family of functions, differential forms or vector fields on M is assumed to be *regular in t*, i.e., depending on t analytically (in the C^ω category) or smoothly (in the C^∞ category).

Proposition 3.1.b. *Proposition 3.1.a holds with the forms ν, μ, vector field Y, and function f replaced by families ν_t, μ_t, Y_t, f_t, $t \in [0,1]$, provided that in the odd-dimensional case either the set $S = \{H = 0\}$ has the extension property (see Section 1) or the families ν_t and Y_t depend on t polynomially.*

This proposition is also proved in Appendix A, using the already mentioned general theorem on division properties.

Remark. Proposition 3.1 also holds for germs at a fixed point.

Another implication of condition (A) concerns the structure of the set $Sing(L)$ of singular points of the characteristic foliation L: it cannot be too degenerate.

Proposition 3.2. *If a Pfaff equation* (ω) *satisfies condition (A) and in the odd-dimensional case the Martinet ideal of* (ω) *has the property of zeros, then any characteristic vector field* X *of* (ω)*, the Martinet hypersurface* S *and the set* $Sing(L)$ *of singular points of the characteristic foliation have the following properties.*

(i) If $n = 2k$*, then the set* $M^n \setminus Sing(L)$ *of points where* X *does not vanish (i.e.* (ω) *is quasi-contact) is dense in* M^n*.*

(ii) If $n = 2k + 1$*, then the set* $M^n \setminus S$ *(i.e., the set of points at which* ω *is contact) is dense in* M^n*. Equivalently, any generator* H *of the Martinet ideal is not a zero divisor.*

(iii) If $n = 2k + 1$*, then the set* $S \setminus Sing(L)$ *is dense in* S*.*

Proof. Statement (i) follows from the observation that if ω is not quasi-contact at any point of an open set, then any characteristic vector field X vanishes on this set (vanishes on a connected component of M^n, in the analytic category). Consequently, given a point p in this set, the characteristic ideal I_p at p generated by the coefficients of X is trivial and contains no non-zero-divisor. This contradicts assumption (A).

Statement (ii) is a simple implication of the property of zeros of the Martinet ideal; see Definition 1.5 and the examples following it.

To prove (iii), assume that there exists a neighbourhood U in M^n of a point $p \in S$ such that a characteristic vector field X vanishes at any point of the set $U \cap S$. By the property of zeros of the Martinet ideal we obtain that $X_p = 0 \bmod(H_p)$, where the subscript indicates the germ at p. This contradicts assumption (A) at the point p. The proof is complete. $\qquad\square$

Propositions 3.1 and 3.2 imply the possibility of choosing the same characteristic vector field for all Pfaff equations with the same characteristic foliation.

Proposition 3.3. *Let* $P_t = (\omega_t)$*,* $t \in [0, 1]$*, be a family of Pfaff equations on* M^n *satisfying the assumptions of Theorem* 1.1*, if* $n = 2k$*, or of Theorem* 1.3*, if* $n = 2k + 1$*. Then for any family* X_t *of characteristic vector fields of* (ω_t) *we have*

$$X_t = R_t X_0, \qquad\qquad\qquad if \quad n = 2k, \quad or$$

$$X_t = R_t X_0 \mod (H), \qquad if \quad n = 2k + 1,$$

where R_t*,* $t \in [0, 1]$*, is a family of positive-valued functions and* X_0 *is* X_t *with* $t = 0$*.*

Proof. Let $n = 2k$. The equality $L(\omega_t) = L(\omega_0)$ implies that $(X_t \wedge X_0)(p) = 0$ for all $p \in M^n$. By Proposition 3.1.b we obtain that $X_t = R_t X_0$, where R_t is a family of functions. Proposition 3.1.b also implies that for any fixed t we have $X_0 = Q_t X_t$, where Q_t is a function on M^n. This leads to the relation $(1 - R_t Q_t) X_0 = 0$. By Proposition 3.2, X_0 does not vanish on a dense subset of M^n; thus $R_t Q_t = 1$. This implies that R_t is a family of nonvanishing functions. This family is positive valued, since for any $p \in M^n$ the function $R_t(p)$ is continuous in t and $R_0(p) = 1$.

In the case of $n = 2k + 1$ the equality $L(\omega_t) = L(\omega_0)$ gives $(X_t \wedge X_0)(p) = 0$ for all $p \in S$. From the property of zeros of the Martinet ideal we deduce that $X_t \wedge X_0 = 0 \bmod(H)$. Using Proposition 3.1.b, we see that $X_t = R_t X_0 \bmod(H)$, for a family of functions R_t. Similarly, we have $X_0 = Q_t X_t \bmod(H)$ for any fixed t, where the Q_t are functions. Therefore, $(1 - R_t Q_t) X_0 = 0 \bmod(H)$. By Proposition 3.2, $R_t Q_t = 1$ on S, and so R_t is nonvanishing on S. From the fact that $R_t(p)$ is continuous in t and from $R_0(p) = 1$, we deduce that R_t is positive valued on S,

for any $t \in [0,1]$. Finally, adding to R_t the function CH^2 with a sufficiently large constant C, we obtain R_t positive valued on M. $\square$

Condition (A) implies one more division property that we need in our proofs. Its proof is postponed to Section 4.

Proposition 3.4. *Let $P_t = (\omega_t)$, $t \in [0,1]$, be a family of Pfaff equations on M^{2k} satisfying the assumptions of Theorem 1.1, and let β_t be a family of 1-forms such that $\omega_t \wedge \beta_t = 0$. Then $\beta_t = h_t \omega_t$ for some family h_t of functions.*

Note that this statement is trivial if ω_t is a family of nonvanishing 1-forms, but we do not assume this in our theorems.

4. Auxiliary algebraic lemmas

To prove the solvability of the homotopy equation in the proofs of Theorems 1.1 and 1.3, we will also use the following simple algebraic facts.

Recall that a 1-form α on M^n, $n \geq 3$, is called contact (quasi-contact) at $p \in M^n$ if $n = 2k + 1$ (respectively, $n = 2k$) and $(\alpha \wedge (d\alpha)^k)(p) \neq 0$ (respectively, $(\alpha \wedge (d\alpha)^{k-1})(p) \neq 0$).

Lemma 4.1. *Let α and λ be 1-forms on M^{2k}. If α is quasi-contact at p and $\left(\lambda \wedge \alpha \wedge (d\alpha)^{k-2}\right)(p) = 0$, then $(\lambda \wedge \alpha)(p) = 0$.*

Lemma 4.2. *Let α be a 1-form on M^{2k+1}. If α is a contact at p and λ is a 1-form such that $\left(\lambda \wedge \alpha \wedge (d\alpha)^{k-1}\right)(p) = 0$ and $\left(\lambda \wedge (d\alpha)^k\right)(p) = 0$, then $\lambda(p) = 0$.*

The facts stated in these lemmas are invariant with respect to multiplication of α by a nonvanishing function, i.e., they are properties of the Pfaff equation (α). These properties can be easily checked in the Darboux coordinates in which the Pfaff equation takes the form $(dz + x_1 dy_1 + \cdots + x_r dy_r)$, where $r = k - 1$ if $n = 2k$, and $r = k$ if $n = 2k + 1$.

Lemma 4.3. *Let α be a 1-form on M^{2k+1} that is not contact at p, but $\alpha(p) \neq 0$. If λ is a 1-form such that $(\lambda \wedge \alpha \wedge (d\alpha)^{k-1})(p) = 0$, then $(\lambda \wedge (d\alpha)^k)(p) = 0$.*

Proof. We take a nonzero vector $v \in T_p M^{2k+1}$ such that $v \rfloor \alpha = v \rfloor d\alpha = 0$. (The existence of such a vector follows from the assumption that α is not contact at p.) Then the relation assumed in the lemma implies that the form $(v \rfloor \lambda) \cdot \alpha \wedge (d\alpha)^{k-1}$ vanishes at p. It follows that if $(\alpha \wedge (d\alpha)^{k-1})(p) \neq 0$, then $(v \rfloor \lambda)(p) = 0$, and consequently $(\lambda \wedge (d\alpha)^k)(p) = 0$. On the other hand, if $(\alpha \wedge (d\alpha)^{k-1})(p) = 0$, then the assumption $\alpha(p) \neq 0$ implies that $(d\alpha)^k(p) = 0$, and then again $(\lambda \wedge (d\alpha)^k)(p) = 0$. $\square$

Lemma 4.4. *If Ω is a volume form on M^n, λ is a 1-form, γ is an $(n-2)$-form and X is a vector field defined by the relation $X \rfloor \Omega = \lambda \wedge \gamma$, then $X \rfloor \lambda = 0$.*

Proof. To prove this statement, note that the definition of X implies $X \rfloor (\lambda \wedge \gamma) = 0$, and consequently $(X \rfloor \lambda) \cdot \gamma \pm (X \rfloor \gamma) \wedge \lambda = 0$. It follows that $(X \rfloor \lambda) \cdot (\lambda \wedge \gamma) = 0$. Since X vanishes exactly at points at which the form $\lambda \wedge \gamma$ vanishes, we obtain that $X \rfloor \lambda = 0$. $\square$

Finally, we need the following general properties of a characteristic vector field.

Lemma 4.5. *If X is a characteristic vector field of a Pfaff equation (ω) on M^{2k}, then*

$$X \rfloor \omega = 0 \quad and \quad (X \rfloor d\omega) \wedge \omega = 0.$$

Proof. The first relation follows from the definition of X and Lemma 4.4. The definition of X implies that $X \rfloor (\omega \wedge (d\omega)^{k-1}) = 0$, which, together with $X \rfloor \omega = 0$, gives $(X \rfloor d\omega) \wedge \omega \wedge (d\omega)^{k-2} = 0$ if $k > 1$. Now the second relation in Lemma 4.5 follows from Lemma 4.1 at points where ω is quasi-contact. At all other points the field X vanishes, and there is nothing to prove. $\square$

Lemma 4.6. *If X is a characteristic vector field of a Pfaff equation (ω) on M^{2k+1} whose Martinet ideal has the property of zeros, and H is a generator of this ideal, then*

$$X \rfloor \omega = 0 \mod (H), \qquad X \rfloor dH = 0 \mod (H), \qquad (X \rfloor d\omega) \wedge \omega = 0 \mod (H).$$

Proof. Due to the property of zeros of (H), it suffices to prove the three relations at any point p of the Martinet hypersurface S such that $X(p) \neq 0$. The relation $(X \rfloor dH)(p) = 0$ follows immediately from the definition of X. To see the other two relations, note that S is regular in a neighbourhood of a point p such that $X(p) \neq 0$. The definition of the characteristic vector field X in the case $n = 2k+1$ implies that the vector field $X|_S$ on S is, in a neighbourhood of such a point p, a characteristic vector field of the Pfaff equation $(\omega|_S)$ on S (which is quasi-contact at p). Thus the remaining two relations follow from Lemma 4.5. The proof is complete. $\square$

Proof of Proposition 3.4. Let X_t be the characteristic vector field of (ω_t) defined by $X_t \rfloor \Omega = \omega_t \wedge (d\omega_t)^{k-1}$. Since X_t (and so ω_t) does not vanish on a dense subset of M^n, the condition $\omega_t \wedge \beta_t = 0$ and Lemma 4.5 imply that $X_t \rfloor \beta_t = 0$ and $X_t \rfloor (\beta_t \wedge (d\omega_t)^{k-1}) = 0$. From Proposition 3.3 we have the equality $X_t = R_t X_0$, with R_t nonvanishing; thus $X_0 \rfloor (\beta_t \wedge (d\omega_t)^{k-1}) = 0$. Therefore the division property in Proposition 3.1.b implies the following relation: $\beta_t \wedge (d\omega_t)^{k-1} = X_0 \rfloor \mu_t = (g_t/R_t) X_t \rfloor \Omega$, where Ω is a volume form, g_t is a family of functions and $\mu_t = g_t \Omega$. Taking $h_t = g_t/R_t$, we can rewrite this relation in the form

$$(\beta_t - h_t \omega_t) \wedge (d\omega_t)^{k-1} = 0.$$

Let us show that this relation implies $\beta_t - h_t \omega_t = 0$. We know that $(\beta_t - h_t \omega_t) \wedge \omega_t = 0$, since $\beta_t \wedge \omega_t = 0$. Fix t and a point p at which ω_t is quasi-contact. At this point ω_t does not vanish; therefore $(\beta_t - h_t \omega_t)(p) = r \omega_t(p)$, with the scalar r depending on t and p. Then the displayed relation implies that $r(\omega_t \wedge (d\omega_t)^{k-1})(p) = 0$ and consequently $r = 0$. So, $(\beta_t - h_t \omega_t)(p) = 0$ if p is a point at which ω_t is quasi-contact. By Proposition 3.2, (i) the set of such points is dense, and so $\beta_t = h_t \omega_t$ at any point of the manifold. The proof is complete. $\square$

Now we are ready to prove the solvability of the homotopy equation (HE) and our main theorems.

5. Proof of Theorem 1.1

Solvability of the homotopy equation (HE) in Section 1 is equivalent to solvability of

$$(5.1) \qquad\qquad \left(L_{Z_t} \omega_t + \frac{d\omega_t}{dt} \right) \wedge \omega_t \wedge (d\omega_t)^{k-2} = 0,$$

with respect to a family Z_t of vector fields. Namely, equation (5.1) is obtained from the homotopy equation by external multiplication by $\omega_t \wedge (d\omega_t)^{k-2}$. Conversely, if (5.1) is solvable then, using the fact that the set of quasi-contact points of (ω_t) is dense in M^n (Proposition 3.2,(i)), we get from (5.1) by Lemma 4.1 that $(L_{Z_t}\omega_t + (d\omega_t/dt))\wedge\omega_t = 0$. Therefore, by Proposition 3.4 we get $L_{Z_t}\omega_t + (d\omega_t/dt) + h_t\omega_t = 0$, for a family of functions h_t, which is the homotopy equation (HE).

A solution Z_t of equation (5.1) will be constructed within the set of families Z_t satisfying

$$(5.2) \qquad\qquad\qquad Z_t \rfloor \omega_t = 0.$$

Condition (5.2) implies that $L_{Z_t}\omega_t = Z_t \rfloor d\omega_t$, and the equation (5.1) can be rewritten in the form

$$(5.3) \qquad Z_t\rfloor \left(\omega_t \wedge (d\omega_t)^{k-1}\right) + (k-1)\frac{d\omega_t}{dt} \wedge \omega_t \wedge (d\omega_t)^{k-2} = 0.$$

In order to solve equation (5.3) we fix a volume form Ω and define a family X_t of characteristic vector fields of (ω_t) by the relation $X_t \rfloor \Omega = \omega_t \wedge (d\omega_t)^{k-1}$. Lemma 4.5 and Proposition 3.3 imply the relations $X_0\rfloor\omega_t = 0$, $X_0\rfloor(d\omega_t/dt) = 0$ and $(X_0\rfloor d\omega_t) \wedge \omega_t = 0$. Thus

$$X_0\rfloor\nu_t = 0, \quad \text{where} \quad \nu_t = \left(\frac{d\omega_t}{dt} \wedge \omega_t \wedge (d\omega_t)^{k-2}\right).$$

Therefore, by the division property in Proposition 3.1.b, we have

$$(5.4) \qquad\qquad\qquad \nu_t = X_0 \rfloor \tilde{\mu}_t,$$

with some family $\tilde{\mu}_t$ of $(n-1)$-forms of the same regularity with respect to t as in ω_t. Using Proposition 3.3 again, we obtain

$$(5.5) \qquad\qquad\qquad \nu_t = X_t \rfloor \mu_t$$

for some, regular in t, family of $(n-1)$-forms μ_t. This relation allows to rewrite equation (5.3) in the form

$$Z_t \rfloor (X_t \rfloor \Omega) + (k-1)X_t \rfloor \mu_t = 0.$$

The latter equation has a solution Z_t defined by the relation

$$Z_t \rfloor \Omega = (k-1)\mu_t.$$

It now remains to check that the constructed solution Z_t satisfies relation (5.2). The equality (5.2) is equivalent to the relation $\mu_t \wedge \omega_t = 0$. From (5.5) and the definition of ν_t we have $(X_t \rfloor \mu_t) \wedge \omega_t = 0$. By Lemma 4.5, $X_t\rfloor\omega_t = 0$; therefore $X_t \rfloor (\mu_t \wedge \omega_t) = 0$. Thus the n-form $\mu_t \wedge \omega_t$ vanishes at any point at which X_t does not vanish. By Proposition 3.2, (i) the set of such points is everywhere dense; therefore $\mu_t \wedge \omega_t = 0$ and (5.2) holds. This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

We will use the following proposition (its proof is postponed to the end of this section).

Proposition 6.1. *Assume that $P_0 = (\omega_0)$ satisfies condition (A) and*

$$(6.1) \qquad\qquad \omega_1 \wedge (d\omega_1)^{k-1} = \omega_0 \wedge (d\omega_0)^{k-1}.$$

Then for the path $\omega_t = (1-t)\omega_0 + t\omega_1$ we have

$$(6.2) \qquad\qquad \omega_t \wedge (d\omega_t)^{k-1} = A_t\,\omega_0 \wedge (d\omega_0)^{k-1},$$

where A_t is a family of functions, polynomial in t.

Proof of Theorem 1.2. The equality $L(\omega_0) = L(\omega_1)$ implies $X_0 \wedge X_1 = 0$, where X_0 and X_1 are characteristic vector fields of P_0 and P_1. Condition (A) satisfied for (ω_0) enables us to use the second division property in Proposition 3.1.a, (i) to deduce that $X_1 = RX_0$ and, equivalently,

$$(6.3) \qquad\qquad \omega_1 \wedge (d\omega_1)^{k-1} = R\,\omega_0 \wedge (d\omega_0)^{k-1},$$

where R is a smooth or analytic function. In fact, R is positive valued, which will follow from the closeness of ω_1 to ω_0. Therefore, assuming $R > 0$, we choose the generator

$$\hat{\omega}_1 = R^{1/k}\omega_1,$$

and we have

$$(6.4) \qquad\qquad \hat{\omega}_1 \wedge (d\hat{\omega}_1)^{k-1} = \omega_0 \wedge (d\omega_0)^{k-1}.$$

Let

$$(6.5) \qquad\qquad \omega_t = (1-t)\,\omega_0 + t\,\hat{\omega}_1.$$

To prove Theorem 1.2 it is sufficient to show that the family of Pfaff equations (ω_t) satisfies the assumptions of Theorem 1.1.

The equality (6.4) allows us to use Proposition 6.1 to conclude that the relation (6.2) holds for the path (6.5). It is clear that (6.2) implies that the family (ω_t) satisfies the assumptions (a) and (b) of Theorem 1.1 provided that the functions A_t in (6.2), $t \in [0,1]$, vanish at no point of M^n. This will follow from the assumption on the C^∞-closeness of ω_1 to ω_0 and Theorem B1 in Appendix B. Define a characteristic vector field X_t of (ω_t) by the relation $X_t \rfloor \Omega = \omega_t \wedge (d\omega_t)^{k-1}$, where Ω is a volume form. By (6.2) we have $X_t = A_t X_0$. The C^∞-closeness of ω_1 to ω_0 implies the C^∞-closeness of $X_t, t \in [0,1]$, to X_0. By Theorem B1 the C^∞-closeness of X_t to X_0 in the equality $X_t = A_t X_0$ implies that the function $A_t, t \in [0,1]$, is C^∞-close to 1. Consequently, A_t vanishes at no point of the manifold. The proof of Theorem 1.2 is complete. $\qquad\square$

Proof of Proposition 6.1. Using (6.1), we may assume that the characteristic vector fields X_0 and X_1 of (ω_0) and (ω_1), respectively, are equal. We shall prove that

$$(6.6) \qquad\qquad X_0 \rfloor \left(\omega_t \wedge (d\omega_t)^{k-1}\right) = 0.$$

Having (6.6), we can use assumption (A) and the division property in Proposition 3.1.b (with polynomial dependence in t), which gives $\omega_t \wedge (d\omega_t)^{k-1} = X_0 \rfloor \mu_t$, where μ_t is a volume form. Let $\omega_0 \wedge (d\omega_0)^k = X_0 \rfloor \Omega$, $\mu_t = A_t \Omega$. Then we get (6.2).

To prove (6.6), we note that by Lemma 4.5 we have

$$(6.7) \qquad\qquad X_0 \rfloor \omega_0 = X_0 \rfloor \omega_1 = 0;$$

therefore $X_0 \rfloor \omega_t = 0$. It follows that in order to prove (6.6) it suffices to prove the equality

$$(6.8) \qquad\qquad (X_0 \rfloor d\omega_t) \wedge \omega_t = 0.$$

It is enough to prove the equality (6.8) at any point p such that $X_0(p) \neq 0$. At such a point $\omega_0(p) \neq 0$ and, since $X_0 = X_1$, $\omega_1(p) \neq 0$. From Lemma 4.5 we have

$(X_0 \rfloor d\omega_i) \wedge \omega_i = 0$, $i = 0, 1$. Thus, there are functions h_0 and h_1, defined in a neighbourhood of p, such that in this neighbourhood we have

$$(6.9) \qquad\qquad X_0 \rfloor d\omega_0 = h_0 \omega_0, \quad X_0 \rfloor d\omega_1 = h_1 \omega_1.$$

We will prove below that

$$(6.10) \qquad\qquad\qquad h_1 = h_0.$$

Then from (6.9) we get $X_0 \rfloor d\omega_t = h\omega_t$, where $h = h_0 = h_1$, and so (6.8) holds.

We will show that (6.10) follows from (6.1). We take the Lie derivative of both parts in (6.1) along the vector field X_0. Using the formula $L_X \eta = d(X \rfloor \eta) + X \rfloor d\eta$ for the Lie derivative, we obtain

$$L_{X_0} \left(\omega_0 \wedge (d\omega_0)^{k-1} \right) = d \left(X_0 \rfloor \left(\omega_0 \wedge (d\omega_0)^{k-1} \right) \right) + X_0 \rfloor (d\omega_0)^k$$
$$= 0 + k(X_0 \rfloor d\omega_0) \wedge (d\omega_0)^{k-1} = kh_0 \omega_0 \wedge (d\omega_0)^{k-1},$$

and similarly

$$L_{X_0} \left(\omega_1 \wedge (d\omega_1)^{k-1} \right) = kh_1 \omega_1 \wedge (d\omega_1)^{k-1}.$$

Comparing these equalities and using (6.1) again, we get the required relation (6.10) (since $\omega_0 \wedge (d\omega_0)^{k-1} = \omega_1 \wedge (d\omega_1)^{k-1}$ does not vanish on a dense subset of M). Proposition 6.1 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

7. Proof of Theorem 1.3

Since the Martinet hypersurfaces of $P_t = (\omega_t)$ are the same for all t, the Martinet ideals are the same by the property of zeros. Thus we can fix a generator H of these ideals. The following two propositions will hold under the assumptions of Theorem 1.3. In the propositions all families are regular with respect to t (smooth in the C^∞ category and analytic in the C^ω category).

Proposition 7.1. *There exists a family of vector fields Y_t satisfying the relation*

$$(7.1) \qquad\qquad \left(L_{Y_t} \omega_t + \frac{d\omega_t}{dt} \right) \wedge \omega_t \wedge (d\omega_t)^{k-1} = 0 \quad \mod (H).$$

Proposition 7.2. *Let μ_t be a family of 1-forms such that*

$$(7.2) \qquad\qquad \mu_t \wedge \omega_t \wedge (d\omega_t)^{k-1} = 0 \quad \mod (H).$$

Then the equation

$$(7.3) \qquad\qquad\qquad L_{Z_t} \omega_t + h_t \omega_t = \mu_t$$

has a solution (Z_t, h_t).

The solvability of the homotopy equation (HE) in Section 1 is a direct corollary of these propositions. Namely, we take $\mu_t = -L_{Y_t} \omega_t - d\omega_t/dt$, and then the pair $(\tilde{Z}_t, h_t)$, with $\tilde{Z}_t = Z_t + Y_t$, solves the homotopy equation (HE).

Proof of Proposition 7.1. We fix a volume form Ω and define a family X_t of characteristic vector fields of (ω_t) by the relation $X_t \rfloor \Omega = \omega_t \wedge (d\omega_t)^{k-1} \wedge dH$. From Proposition 3.3 we have

$$X_t = R_t X_0 \quad \mod (H),$$

where R_t is a family of nonvanishing functions, regular in t (of the same regularity in t as in the family ω_t). By Lemma 4.6 we have $X_t \rfloor \omega_t = 0 \mod (H)$ and $(X_t \rfloor d\omega_t) \wedge \omega_t = 0 \mod (H)$. We may replace X_t with X_0 in these equalities. In

particular, we get $X_0 \rfloor \omega_t = 0 \bmod (H)$, which implies $X_0 \rfloor (d\omega_t/dt) = 0 \bmod (H)$. Taking all these equalities into account, we see that

$$(7.4) \qquad X_0 \rfloor \left(\frac{d\omega_t}{dt} \wedge \omega_t \wedge (d\omega_t)^{k-1} \right) = 0 \quad \bmod (H).$$

This equality and Proposition 3.1.b imply that

$$\frac{d\omega_t}{dt} \wedge \omega_t \wedge (d\omega_t)^{k-1} = X_0 \rfloor (f_t \Omega) \quad \bmod (H),$$

where f_t is a family of functions, regular in t. Replacing X_0 with $R_t^{-1} X_t$ and using the definition of X_t, we see that we can rewrite this relation in the form

$$\frac{d\omega_t}{dt} \wedge \omega_t \wedge (d\omega_t)^{k-1} = g_t \, \omega_t \wedge (d\omega_t)^{k-1} \wedge dH \quad \bmod (H),$$

where $g_t = f_t/R_t$. This allows us to rewrite equation (7.1) in the form

$$(7.5) \qquad (L_{Y_t} \omega_t - g_t dH) \wedge \omega_t \wedge (d\omega_t)^{k-1} = 0 \quad \bmod (H).$$

It is clear that (7.5) holds if Y_t satisfies the relations

$$(7.6) \qquad Y_t \rfloor d\omega_t = 0, \quad Y_t \rfloor \omega_t = g_t H,$$

since in this case

$$L_{Y_t} \omega_t = d(Y_t \rfloor \omega_t) = d(g_t H) = g_t dH \quad \bmod (H).$$

Since (H) is the Martinet ideal of (ω_t), we have

$$(7.7) \qquad \omega_t \wedge (d\omega_t)^k = S_t H \, \Omega$$

for a family S_t of nonvanishing functions which has the same regularity in t as in ω_t (this follows from the regularity of the left-hand side and the fact that division by H is a continuous linear operator in the space of smooth functions, see Theorem B1 in Appendix B). Let us show that (7.6) holds for the family Y_t defined by

$$(7.8) \qquad Y_t \rfloor \Omega = \frac{g_t}{S_t} (d\omega_t)^k.$$

In fact, applying $Y_t \rfloor$ to (7.8), we get $g_t (Y_t \rfloor d\omega_t) \wedge (d\omega_t)^{k-1} = 0$. This relation implies $Y_t \rfloor d\omega_t = 0$ (at points where $g_t(p) = 0$ we have $Y_t(p) = 0$, and at other points we can use Lemma 4.2 with $\lambda = (Y_t \rfloor d\omega_t)$ and the fact that contact points are dense). We have shown the first equality in (7.6). In order to prove the second one we apply $Y_t \rfloor$ to (7.7) and, using (7.8), we obtain that $(Y_t \rfloor \omega_t - g_t H) \cdot (d\omega_t)^k = 0$. This implies that $Y_t \rfloor \omega_t - g_t H = 0$ at points where the form $(d\omega_t)^k$ does not vanish, in particular, at points where ω_t is contact. By Proposition 3.2, (ii) the set of such points is everywhere dense; therefore $Y_t \rfloor \omega_t - g_t H = 0$ everywhere, and so (7.6) holds. Proposition 7.1 is proved. $\qquad \square$

Proof of Proposition 7.2. By Lemma 4.2, (i) and the fact that the set of contact points is dense in M^n (Proposition 3.2, (ii)), the equation (7.3) reduces to the following two equations:

$$(7.9) \qquad (L_{Z_t} \omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-1} = \mu_t \wedge \omega_t \wedge (d\omega_t)^{k-1},$$

$$(7.10) \qquad (L_{Z_t} \omega_t + h_t \omega_t) \wedge (d\omega_t)^k = \mu_t \wedge (d\omega_t)^k$$

(with unknown Z_t and h_t), obtained from (7.3) by external multiplication by the forms $\omega_t \wedge (d\omega_t)^{k-1}$ and $(d\omega_t)^k$, respectively.

To solve equation (7.9) we use assumption (7.2). By this assumption

$$(7.11) \qquad \mu_t \wedge \omega_t \wedge (d\omega_t)^{k-1} = H\nu_t$$

for some family ν_t of $2k$-forms, regular in t by Theorem B1 in Appendix B. This permits us to find an explicit solution Z_t of (7.9). Namely, since H is a generator of the Martinet ideal of (ω_t), we have relation (7.7), i.e., $\omega_t \wedge (d\omega_t)^k = HS_t\Omega$, where S_t is a family of nonvanishing functions, regular in t. Let us show that the family of vector fields Z_t defined by the relation

$$(7.12) \qquad Z_t \rfloor \Omega = \frac{k}{S_t}\nu_t$$

is a solution of the equation (7.9). Relation (7.11) and the fact that H is not a zero divisor imply that $\nu_t \wedge \omega_t = 0$. This and (7.12) imply that

$$Z_t \rfloor \omega_t = 0.$$

Consequently, $L_{Z_t}\omega_t = Z_t \rfloor d\omega_t$ and

$$(7.13) \quad (L_{Z_t}\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-1} = (Z_t \rfloor d\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-1} = \frac{1}{k} Z_t \rfloor \left(\omega_t \wedge (d\omega_t)^k\right).$$

Now (7.9) follows from equalities (7.7) and (7.11)–(7.13).

To prove Proposition 7.2 it remains to solve equation (7.10) with respect to h_t. Since $Z_t \rfloor \omega_t = 0$, then $(L_{Z_t}\omega_t) \wedge (d\omega_t)^k = (k+1)^{-1} Z_t \rfloor (d\omega_t)^{k+1} = 0$, and the equation (7.10) takes the form

$$h_t \omega_t \wedge (d\omega_t)^k = \mu_t \wedge (d\omega_t)^k.$$

Due to relation (7.7), to prove that this equation has a solution h_t it suffices to prove that $\mu_t \wedge (d\omega_t)^k = HC_t\Omega$, where C_t is a family of functions, regular in t. We shall first prove that

$$(7.14) \qquad \left(\mu_t \wedge (d\omega_t)^k\right)(p) = 0, \quad \text{for } p \in S.$$

This follows from relation (7.2). Namely, since ω_t is not contact at a point $p \in S$, thus (7.14) follows from (7.2) by Lemma 4.3, provided that $\omega_t(p) \neq 0$. Since the set of points of S at which ω_t vanishes is a subset of the set $Sing(L)$, the set of points $p \in S$ where $\omega_t(p) \neq 0$ is dense in S by Proposition 3.2, (iii). Therefore (7.14) holds at all points $p \in S$. By the property of zeros of the ideal (H) we obtain $\mu_t \wedge (d\omega_t)^k = HC_t\Omega$. Since μ_t and ω_t are regular in t, we deduce from Theorem B1 in Appendix B that C_t is regular in t. Proposition 7.2 is proved. $\qquad \square$

We have completed the proof of Theorem 1.3. Note that the extension property of S was used only when referring to Proposition 3.1.b., and therefore it is not needed if ω_t is polynomial in t (cf. the remark after Theorem 1.3).

8. Proof of Theorem 1.4

Since the Martinet hypersurfaces for $P_0 = (\omega_0)$ and $P_1 = (\omega_1)$ are the same and the Martinet ideals have the property of zeros, they are equal and we can choose a common generator H which will be used throughout the proofs. The following proposition holds under the assumptions of Theorem 1.4 and will enable us to reduce the problem to Theorem 1.3.

Proposition 8.1. *Assume that*

$$\omega_1 \wedge (d\omega_1)^{k-1} \wedge dH = \omega_0 \wedge (d\omega_0)^{k-1} \wedge dH \qquad \mathrm{mod}\ (H).$$

Then, for the path $\omega_t = (1-t)\omega_0 + t\omega_1$ we have

(8.1) $$\omega_t \wedge (d\omega_t)^{k-1} \wedge dH = B_t\, \omega_0 \wedge (d\omega_0)^{k-1} \wedge dH \quad \mathrm{mod}\ (H),$$

and

(8.2) $$\omega_t \wedge (d\omega_t)^k = C_t\, \omega_0 \wedge (d\omega_0)^k,$$

where B_t and C_t are families of functions, polynomial in t.

Proof of Theorem 1.4. Let X_0 and X_1 be characteristic vector fields of P_0 and P_1 defined via the same volume form and the same generator H of the Martinet ideal. Since $L(\omega_1) = L(\omega_0)$, then $X_1 \wedge X_0 = 0 \,\mathrm{mod}\ (H)$. From condition (A) and the division property in Proposition 3.1.a we obtain $X_1 = R X_0 \,\mathrm{mod}\ (H)$ or, equivalently,

(8.3) $$\omega_1 \wedge (d\omega_1)^{k-1} \wedge dH = R\,\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH \quad \mathrm{mod}\ (H),$$

where R is a smooth or analytic function. We will later show, using closeness of ω_1 to ω_0, that R is positive valued. Therefore, we can change the generator of P_1 for

(8.4) $$\hat\omega_1 = R^{1/k}\omega_1,$$

and we get

(8.5) $$\hat\omega_1 \wedge (d\hat\omega_1)^{k-1} \wedge dH = \omega_0 \wedge (d\omega_0)^{k-1} \wedge dH \quad \mathrm{mod}\ (H).$$

Let

(8.6) $$\omega_t = (1-t)\,\omega_0 + t\,\hat\omega_1.$$

To prove Theorem 1.4 it is enough to show that the family of Pfaff equations (ω_t) satisfies the assumptions of Theorem 1.3. Note that we do not need the extension property of S, since the family ω_t in (8.6) is polynomial (in fact, affine) in t, and in this case Theorem 1.3 was proved without using this assumption.

The equality (8.5) enables us to use Proposition 8.1. It is clear that relations (8.1) and (8.2) imply that the family (ω_t) satisfies the assumptions (a), (b) and (c) of Theorem 1.3 provided that the functions B_t do not vanish on S (then the characteristic line field does not change) and C_t do not vanish on M^n (then the Martinet ideal does not change), for $t \in [0,1]$.

The fact that B_t and C_t do not vanish follows from the C^∞-closeness of ω_1 to ω_0 and Theorem B1 in Appendix B. Since ω_1 is C^∞-close to ω_0, then the vector field X_1 is C^∞ close to X_0. The relation $X_1 = R X_0 \,\mathrm{mod}\ (H)$ and continuity of the inverse to the operator $L_{X,H}$ in Theorem B1 imply that there exists a function $\tilde R$ that is C^∞-close to 1 and equal to R at any point of the Martinet hypersurface S. By the property of zeros of the Martinet ideal, $\tilde R = R \,\mathrm{mod}\ (H)$. We may replace R by $\tilde R$ in (8.3) and in the definition (8.4) of $\hat\omega_1$. Then $\omega_t, t \in [0,1]$, is C^∞-close to ω_0. Define a family of characteristic vector fields X_t by the relation $X_t \rfloor \Omega = \omega_t \wedge (d\omega_t)^{k-1} \wedge dH$. Then $X_t, t \in [0,1]$, is C^∞-close to X_0. The equality (8.1) is equivalent to $X_t = B_t X_0 \,\mathrm{mod}\ (H)$. We again use continuity of the inverse to the operator $L_{X,H}$ in Theorem B1. By this theorem there exists a function $\tilde B_t$ that is C^∞-close to 1 and equal to B_t at any point of S. Therefore $B_t > 0$ at any point of S.

To prove that C_t vanishes at no points of M^n, we also use the C^∞-closeness of $\omega_t, t \in [0, 1]$, to ω_0 shown above. Let $\omega_t \wedge (d\omega_t)^k = Q_t H \Omega$, where Ω is a volume form. Relation (8.2) implies that $Q_t = C_t Q_0$. The C^∞-closeness of ω_t to ω_0 implies the C^∞-closeness of the function HQ_t to HQ_0. By continuity of the inverse to the operator $f \to fH$ (Theorem B1) we get the C^∞-closeness of Q_t to Q_0. Since Q_0 is a nonvanishing function, then C_t is C^∞-close to 1. The proof of Theorem 1.4 is complete. $\qquad\square$

Proof of Proposition 8.1. It is enough to prove the equalities

$$(8.7) \qquad X_0 \,\rfloor\, \big(\omega_t \wedge (d\omega_t)^{k-1} \wedge dH\big) = 0 \mod (H),$$

$$(8.8) \qquad X_0 \,\rfloor\, \big(\omega_t \wedge (d\omega_t)^k\big) = 0 \mod (H).$$

Namely, equality (8.7) and condition (A) allow us to use the division properties in Proposition 3.1.b to conclude that $\omega_t \wedge (d\omega_t)^{k-1} \wedge dH = B_t X_0 \rfloor \Omega \mod (H)$, and so the validity of relation (8.1), where B_t is a family of functions, polynomial in t. Equality (8.8) implies that $\omega_t \wedge (d\omega_t)^k$ vanishes at those points of S at which X_0 does not vanish. By Proposition 3.2, (iii) the set of such points is dense in S; therefore $\omega_t \wedge (d\omega_t)^k$ vanishes at all points of S. By the property of zeros of the Martinet ideal we have $\omega_t \wedge (d\omega_t)^k = 0 \mod (H)$, and consequently (8.2) holds for some family of functions C_t, polynomial in t.

In order to prove (8.7) and (8.8) we use the assumption of Proposition 8.1 and choose characteristic vector fields X_0 and X_1 of (ω_0) and (ω_1) equal modulo (H). By Lemma 4.6 we have

$$X_0 \rfloor \omega_i = 0 \mod (H), \qquad X_0 \rfloor dH = 0 \mod (H),$$

for $i = 0, 1$, and therefore $X_0 \rfloor \omega_t = 0 \mod (H)$. It follows that in order to prove (8.7) and (8.8) it suffices to prove the equality

$$(8.9) \qquad (X_0 \rfloor d\omega_t) \wedge \omega_t = 0 \mod (H).$$

Due to the property of zeros of the Martinet ideal, it suffices to prove this equality for any point $p \in S$ such that $X_0(p) \neq 0$. At such points the 1-form dH does not vanish and S is smooth. Since $X_1 = X_0 \mod (H)$, using Lemma 4.6 we obtain

$$(X_0 \rfloor d\omega_0) \wedge \omega_0 = (X_0 \rfloor d\omega_1) \wedge \omega_1 = 0$$

in a neighbourhood of p in S. Since $X_0(p) \neq 0$, then $\omega_0(p) \neq 0$ and $\omega_1(p) \neq 0$, and therefore these relations imply the equalities

$$(8.10) \qquad X_0 \rfloor d\omega_0 = h_0 \omega_0, \qquad X_0 \rfloor d\omega_1 = h_1 \omega_1,$$

which hold in a neighbourhood U of p in S. Here h_0 and h_1 are functions defined in this neighbourhood. We will show that $h_0 = h_1$; then (8.10) implies that $X_0 \rfloor d\omega_t = h_0 \omega_t$, and (8.9) holds in the neighbourhood U.

To prove that $h_0 = h_1$ on $U \subset S$, we restrict the relation assumed in Proposition 8.1 to the tangent bundle of U. We obtain $(\omega_1 \wedge d\omega_1)|_U = (\omega_0 \wedge (d\omega_0)^{k-1})|_U$ and take the Lie derivative of this relation along the restriction $X_0|_S$ of X_0 to S (recall that X_0 is tangent to S). As in the proof of Proposition 6.1, we obtain the required equality $h_0 = h_1$. Proposition 8.1 is proved. $\qquad\square$

Appendix A. Division properties

In this Appendix we present a general theorem on division properties of the exterior (respectively, interior) product with a section X of a vector bundle. This theorem is proved in [DJ] and implies our Propositions 3.1.a and 3.1.b. Our results hold in the categories C^s, where $s = \infty$ or $s = \omega$.

Let M be a paracompact differential manifold. Consider a vector bundle E over M of rank m and denote by E^* its dual bundle. Let $\Lambda_r = \Lambda_r(E)$ denote the rth exterior power of E, $r = 0, 1, \ldots, m$, with $\Lambda_0 = M \times \mathbb{R}$ and $\Lambda_1 = E$. We denote by $\Lambda_r(M; E)$ the linear space of sections of Λ_r (smooth or real analytic, depending on the category).

Any section α of E defines the linear operator of exterior multiplication by α, which gives the complex

(A.1) $$0 \to \Lambda_0(M) \to \Lambda_1(M; E) \to \cdots\cdots \to \Lambda_m(M; E),$$

with the operator $\partial_\alpha = \partial_\alpha^p : \Lambda_p(M; E) \to \Lambda_{p+1}(M; E)$ defined by $\partial_\alpha^p(\gamma) := \alpha \wedge \gamma$.

Consider a section X of the dual bundle E^*. This section defines the operator of the interior product with X, $X \rfloor : \Lambda_r(M; E) \to \Lambda_{r-1}(M; E)$. Given a local basis $e_1, \ldots, e_m$ of E, the operator of the interior product with X is defined on the elements of a local basis of Λ_r by

$$X \rfloor (e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{j=1}^m (-1)^{j-1} \langle X, e_{i_j} \rangle \, e_{i_1} \wedge \cdots \breve{e}_{i_j} \cdots \wedge e_{i_m},$$

where $\breve{e}_{i_j}$ means absence of e_{i_j} and $\langle \cdot, \cdot \rangle$ denotes the duality product between E^* and E. Clearly, $(X \rfloor)^2 = 0$; so the operator $X \rfloor$ defines the complex

(A.2) $$0 \to \Lambda_m(M; E) \to \Lambda_{m-1}(M; E) \to \cdots \to \Lambda_1(M; E) \to \Lambda_0(M; E).$$

Let S be a closed subset of M. Denote by $\Lambda_r(M, S; E) \subset \Lambda_r(M; E)$ the subspace of sections of $\Lambda_r(M; E)$ vanishing at all points of S, and let

$$\Lambda_r(S; E) = \Lambda_r(M; E)/\Lambda_r(M, S; E)$$

denote the quotient space.

Any element α of $\Lambda_1(S; E)$ defines the unique operator

$$\partial_\alpha^p : \Lambda_p(S; E) \to \Lambda_{p+1}(S; E)$$

(the quotient of the operator of exterior multiplication), which gives the complex

(A.3) $$0 \to \Lambda_0(S) \to \Lambda_1(S; E) \to \cdots\cdots \to \Lambda_m(S; E).$$

Given a section X of E^*, the operator $X \rfloor$ defines the following complex on the quotient spaces:

(A.4) $$0 \to \Lambda_m(S; E) \to \Lambda_{m-1}(S; E) \to \cdots \to \Lambda_1(S; E) \to \Lambda_0(S; E).$$

We define the invariant $d_p(X) = \operatorname{depth}(I_p)$, where I_p is the ideal of function germs at $p \in M$ generated by the coefficients $a_1, \ldots, a_m$ of X in a local basis of E^* (cf. Definition 1.2 in Section 1). Similarly, given a pair (H, X) of a function H and a section X of E^* on M, we define $d_p(H, X)$ as the maximal length of a regular sequence of function germs that begins with the germ H_p of H at p and has further elements in I_p (cf. Definition 1.4). Analogously we define the invariants $d_p(\alpha)$ and $d_p(H, \alpha)$.

Statements (i) and (ii) of the following theorem hold in the C^∞ and C^ω categories, for $0 \le q \le n-1$.

Theorem A. *(i) If α satisfies the condition $d_p(\alpha) \ge q+1$ for all $p \in M$ such that $\alpha(p) = 0$, then the complex (A.1) is exact up to $\Lambda_q(M; E)$. Similarly, if $d_p(X) \ge q+1$ for all $p \in M$ such that $X(p) = 0$, then the complex (A.2) is exact up to $\Lambda_{m-q}(M; E)$.*

(ii) Let H be a function on M such that the ideal (H) has the property of zeros, and let $S = \{H = 0\}$. If α is a section of E on M such that (H, α) satisfies $d_p(H, \alpha) \ge q+2$ for all $p \in S$ such that $\alpha(p) = 0$, then the complex (A.3) is exact up to $\Lambda_q(S; E)$. Similarly, if X is a section of E^* on M and $d_p(H, X) \ge q+2$ for all $p \in S$ such that $X(p) = 0$, then the complex (A.4) is exact up to $\Lambda_{m-q}(S; E)$.

(iii) If the assumptions of (i) hold, then, in the C^∞ category, the complex (A.1) splits up to $\Lambda_{q-1}(M; E)$ and the complex (A.2) splits up to $\Lambda_{m-q+1}(M; E)$. If the assumptions of (ii) hold and S has the extension property, then the complex (A.3) splits up to $\Lambda_{q-1}(S; E)$ and the complex (A.4) splits up to $\Lambda_{m-q+1}(S; E)$. Here the corresponding spaces are equipped with the C^∞ topology and are considered as Fréchet spaces (quotient Fréchet spaces).

Above, a complex $0 \to L_m \to \cdots \to L_{m-q+1} \to L_{m-q} \to \cdots$ defined by the operators $\partial_i : L_i \to L_{i-1}$ is called exact up to L_{m-q} if $\operatorname{Im} \partial_{i+1} = \ker \partial_i$ for $i = m, m-1, \ldots, m-q$, and it splits up to L_{m-q+1} if the L_i are linear topological spaces, $\operatorname{Im} \partial_i$ are closed subspaces of L_{i-1} and each $\partial_i : L_i \to L_{i-1}$ has a continuous right inverse K_i defined on $\operatorname{Im} \partial_i$, for all $i = m, m-1, \ldots, m-q+1$.

The above theorem follows from Theorems 2.1 and 2.2 in [DJ]. In the local case (of germs) statements (i) and (ii) follow from a well-known algebraic result on exactness of the Koszul complex, cf. e.g. [E] or [JZh2], Appendix 1.

Proof of Proposition 3.1.a. In the even-dimensional case the first implication follows trivially from statement (i) in Theorem A if we take the bundle E equal to the cotangent bundle $E = T^*M$, the dual $E^* = TM$, and consider the complex (A.2). ($\Lambda_r(M, E)$ is identified with the space of differential r-forms on M.) The second implication follows analogously from the same statement by taking $E = TM$ and the complex (A.1).

In the odd-dimensional case the first implication follows in a similar way from statement (ii) in Theorem A concerning the complex (A.4). This is because the property of zeros of (H) allows us to identify the elements of $\Lambda_r(S; E)$ with the equivalence classes of differential r-forms modulo (H) (cf. our convention on notation mod (H) presented after Definition 1.3). The second implication follows analogously from statement (ii) in Theorem A concerning the complex (A.3).   □

Proof of Proposition 3.1.b. The existence of μ_t and f_t for any fixed t follows from Proposition 3.1.a. We have to show the regularity of these families in t. We shall prove the regularity of μ_t (the proof of regularity of f_t is analogous). If ν_t depends on t polynomially, then Proposition 3.1 allows us to construct μ_t polynomial in t, and the regularity follows trivially. In the general case our arguments are different for the categories C^ω and C^∞.

In the C^ω category we use the following fact: if a sequence $a_1, \ldots, a_r$ of real analytic function germs at $p \in M$ is regular in the ring of analytic function germs at p, then it is regular when considered as a sequence in the ring of real analytic function germs of the variables $(x, t) \in M \times \mathbb{R}$ at (p, t_0), for any $t_0 \in [0, 1]$. Using

local coordinates, this fact can be easily proved for the case of formal power series using the definition of regular sequence in the ring of formal power series of the variables $x_1, \ldots, x_n, t$. Then, using the fact that the ring of formal power series is faithfully flat over the ring of convergent series (see Malgrange [Mlg], Chapter 3), we see that it also holds for converging series and so for germs of analytic functions. Using the above fact we see that the assumption (A) holds over the manifold $\tilde{M} = M \times I$, where I is an open interval containing $[0, 1]$ on which the analytic family ν_t is well defined by analytic extension. Thus we can use Theorem A over the manifold $\tilde{M}$, i.e., for the bundles $E = T^*M$ and $E^* = TM$ pulled back to $\tilde{M}$ by the canonical projection $M \times I \to M$.

In the C^∞ category, in the even-dimensional case the smooth dependence of μ_t on t follows from statement (iii) in Theorem A. By this statement there exists a continuous right inverse operator K to the linear operator $X \rfloor : \Lambda_{r+1}(M; E) \to \Lambda_r(M; E)$, for $r = n-1$ and $r = n-2$, and we can define $\mu_t = K\nu_t$. Here $E = T^*M$ and $\Lambda_r(M; E) = \Lambda_r(M)$, the space of differential r-forms on M.

In the C^∞ category, in the odd-dimensional case we also use statement (iii) of Theorem A and the extension property of S. Namely, for $E = T^*M$ we define $\mu_t = \lambda K\nu_t|_S$, where $K : \Lambda_{n-1}(S; E) \to \Lambda_n(S; E)$ is the continuous right inverse operator to $X \rfloor : \Lambda_n(S; E) \to \Lambda_{n-1}(S; E)$, and $\lambda : \Lambda_n(S; E) \to \Lambda_n(M; E)$ is a continuous linear operator of extension. $\qquad\square$

Appendix B. Continuity of division

Continuity of division in the cases presented below is needed in the main proofs and will be proved separately. Let $C^\infty(M)$ and $Vect^\infty(M)$ be the spaces of smooth functions and smooth vector fields on M, with the C^∞ topology. Let $C^\infty(M, S)$ and $Vect^\infty(M, S)$ be the subspaces of functions (vector fields) on M vanishing on the Martinet hypersurface $S \subset M$. The quotient Fréchet spaces

$$C^\infty(S) = C^\infty(M)/C^\infty(M, S), \qquad Vect^\infty(S; TM) = Vect^\infty(M)/Vect^\infty(M, S)$$

can be identified with the space of smooth functions on S and the space of smooth sections of the tangent bundle TM restricted to S, respectively.

Given a Pfaff equation on M^{2k} and a characteristic vector field X, we consider the linear operator

$$L_X : \ C^\infty(M) \to Vect^\infty(M), \qquad L_X(f) = fX.$$

For a Pfaff equation on M^{2k+1}, a characteristic vector field X and a generator H of the Martinet ideal we consider the linear operators

$$L_H : \ C^\infty(M) \to C^\infty(M), \qquad L_H(f) = Hf,$$

and

$$L_{X,H} : \ C^\infty(S) \to Vect^\infty(S; TM), \qquad L_{X,H}([f]) = [fX],$$

where $[\,\cdot\,]$ denotes the equivalence class in the corresponding quotient space.

Theorem B1. *If the characteristic vector field X satisfies condition (A) and in the odd-dimensional case the Martinet ideal (H) has the property of zeros, then each of the linear operators L_X, L_H and $L_{X,H}$ is injective, has closed image, and has continuous inverse defined on the image.*

Proof. By the Banach open mapping theorem in Fréchet spaces it suffices to prove that each of the operators L_X, L_H, and $L_{X,H}$ is injective and has closed image.

The injectivity of the operators L_X, L_H and $L_{X,H}$ follows from Proposition 3.2, (i), (ii), (iii), respectively. The closedness of the image of the operator L_H follows from the global property of zeros implied by Definition 1.5—by this property the image of L_H coincides with the closed subspace $C^\infty(M, S) \subset C^\infty(M)$ of functions vanishing on S.

To prove the closedness of the image of the operators L_X and $L_{X,H}$, we use Proposition 3.1. By Proposition 3.1, (i) the image of L_X coincides with the kernel of the continuous operator $Vect^\infty(M) \to \Gamma^\infty(\Lambda^2 TM)$ defined by $Y \to X \wedge Y$, where $\Gamma^\infty(\Lambda^2 TM)$ is the space of smooth sections of the skew-symmetric product of the tangent bundle TM, with the C^∞ topology. Similarly, by Proposition 3.1, (ii) the image of $L_{X,H}$ coincides with the kernel of the continuous operator $Vect^\infty(S) \to \Gamma^\infty(\Lambda^2(S; TM))$ given by $[Y] \to [X \wedge Y]$, where $\Gamma^\infty(\Lambda^2(S; TM))$ is the space of smooth sections over S of the skew-symmetric product of the tangent bundle TM (with the C^∞ topology) and $[\,\cdot\,]$ denotes the equivalence class in the corresponding quotient space. The kernel of this operator is a closed subspace of $Vect^\infty(S; TM) = Vect^\infty(M)/Vect^\infty(M, S)$. The proof is complete. $\square$

It is natural to ask if it is possible to replace the C^∞-closeness of ω_1 to ω_0 in Theorems 1.2 and 1.4 by C^r-closeness with some r. Any attempt at answering this question requires modification of Theorem B1, which was used in the proofs of Theorems 1.2 and 1.4. Proving Theorem 1.2, we had to show that the function A_t, $t \in [0,1]$, does not vanish at any point of M. In the proof of Theorem 1.4 we had to show that the functions B_t, $t \in [0,1]$, do not vanish at points of S and the functions C_t, $t \in [0,1]$, do not vanish at points of M. The C^∞-closeness of ω_1 to ω_0 given as an assumption in Theorems 1.2 and 1.4 and the continuity of the inverse to the operators L_X, L_H and $L_{X,H}$ allowed us to obtain the C^∞-closeness of A_t, B_t and C_t to 1. Of course, to conclude that these functions do not vanish, their C^0-closeness to 1 would be enough.

We introduce the following topological characteristic of a linear injective operator $L : C^\infty(M) \to C^\infty(M)$ or $L : C^\infty(M) \to Vect^\infty(M)$ or $L : C^\infty(S) \to Vect^\infty(S)$. Denote by $m \in \{0, 1, 2, \ldots; \infty\}$ the minimal m such that for any $s \geq 0$ the convergence to 0 of the sequence of sections $L(f_n)$ in the C^{s+m} topology implies the convergence to 0 of the sequence of functions f_n in the C^s topology. This means that the inverse to L behaves not worse than a linear differential operator of order m. Note that by Theorem B1 we have $m(L_X)$, $m(L_H)$, and $m(L_{X,H}) \leq \infty$.

In many cases the numbers $m(L_X), m(L_H)$ and $m(L_{X,H})$ are finite and can be found or estimated from above, see examples below. Tracing the construction of the functions A_t, B_t and C_t in the proofs of Theorems 1.2 and 1.4, it is easy to check that if these numbers are finite, then:

1. the C^0-closeness of A_t to 1 holds provided that the 1-form ω_1 is close to ω_0 in the C^r topology with $r = 2m(L_X) + 2$;

2. the C^0-closeness of B_t to 1 holds provided that the 1-form ω_1 is close to ω_0 in the C^r topology with $r = 2m(L_{X,H}) + 2$;

3. the C^0-closeness of C_t to 1 holds provided that the 1-form ω_1 is close to ω_0 in the C^r topology with $r = m(L_{X,H}) + m(L_H) + 2$.

Therefore in Theorems 1.2 and 1.4 the C^∞-closeness of ω_1 to ω_0 can be replaced by the closeness in a weaker topology, and we obtain the following result.

Theorem B2. *In Theorem* 1.2 *the C^∞-closeness of ω_1 to ω_0 can be replaced by the C^r-closeness with $r = 2m(L_X) + 2$. In Theorem* 1.4 *the C^∞-closeness of ω_1 to ω_0 can be replaced by the C^r-closeness with*

$$r = \max\left(2m(L_{X,H}) + 2, m(L_{X,H}) + m(L_H) + 2\right).$$

Examples ($n = 2k$)**.** 1. If X has no singular points, then it is clear that $m(L_X) = 0$. Therefore the C^∞-closeness of ω_1 to ω_0 in Theorem 1.2 can be replaced by C^2-closeness. We obtain Theorem 0.1.

2. If the 1-jet of X vanishes at no points of the manifold, then it is easy to prove that $m(L_X) \le 1$. Therefore the C^∞-closeness of ω_1 to ω_0 in Theorem 1.2 can be replaced by C^4-closeness.

Examples ($n = 2k + 1$)**.** 1. If (ω_0) is a Martinet distribution, i.e., $dH(p) \ne 0$ and $X(p) \ne 0$ for any $p \in S = \{H = 0\}$, then it is easy to prove that $m(L_H) \le 1$ and $m(L_{X,H}) = 0$. Therefore the C^∞-closeness of ω_1 to ω_0 in Theorem 1.4 can be replaced by C^3-closeness. We obtain Theorem 0.2.

2. Assume that $dH(p) \ne 0$ for any $p \in S = \{H = 0\}$. Then the restriction of X to S is a smooth vector field $X|_S$ on S. Assume that the 1-jet of X_S does not vanish. In this case $m(L_H) \le 1$ and $m(L_{X,H}) \le 1$. Therefore the C^∞-closeness of ω_1 to ω_0 in Theorem 1.4 can be replaced by C^4-closeness.

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